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Stationary Solutions in Three-Dimensional General Relativity

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All the stationary solutions of the three-dimensional vacuum Einstein equations are obtained. These include a class of multicenter solutions representing systems of massive and spinning point particles. The geodesic motion of a test particle in the one-particle metric is discussed. A class of geodesics contain finite intervals where the particle moves back in coordinate time, without violation of causality.

1. INTRODUCTION

In three space-time dimensions, the Riemann tensor is uniquely determined by the Ricci tensor, so that any solution of the vacuum Einstein equations is flat, except for possible singularities (point particles). Because of the absence of geodesic deviation between particles (Deser et al., 1984), static multiparticle systems are possible and may easily be constructed (Deser et al., 1984; Clément, 1976, 1983). This situation is curiously similar to that of five-dimensional general relativity, where the vacuum Einstein equations have been shown to admit static multiparticle solutions[2] in the axisymmetric case (Clément, 1984). It may therefore be worthwhile to study more closely the three-dimensional case, as a model for the more complicated five-dimensional case.

The one-particle rotating solution of three-dimensional general relativity was also constructed in Deser et al. (1984). In this work we shall study in more detail the stationary solutions. All the stationary solutions of the three-dimensional vacuum Einstein equations are derived in Section 2 of this paper. A particularly interesting class is that of multicenter solutions representing systems of massive and spinning point particles. The effect of

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[2]These solutions are regular, so that the corresponding particles are extended in the three space dimensions.

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the one-particle metric on the geodesic motion of a test particle is investigated in Section 3. A class of geodesics is found to contain finite intervals where the particle appears to move back in time. This peculiarity does not conflict with causality.

2. GENERAL STATIONARY SOLUTION

The general stationary metric can always be written (Landau and Lifshitz, 1970)

$$ds^2 = h^2(dt + \omega_i\, dx^i)^2 + \bar{g}_{ij}\, dx^i\, dx^j \tag{1}$$

where the metric functions h, ω_i, $\bar{g}_{ij}$ depend only on the space coordinates x^i. The arbitrariness of the vector potential ω_i under gravitational gauge transformations $t \to t + F(\mathbf{r})$ can be restricted by the gauge condition

$$\omega^i_{\ ;i} = 0 \tag{2}$$

(in this and the following equations, the spatial metric $\bar{g}_{ij}$ is used to raise and lower indices and to define the covariant derivative). The vacuum Einstein equations may then be written (Landau and Lifshitz, 1970)

$$\begin{aligned} R_{00} &\equiv -hh^{;i}_{;i} + \frac{h^4}{4} f^{ij} f_{ij} = 0 \\ R^i_0 &\equiv -\tfrac{1}{2} h^{-1} (h^3 f^{ij})_{;j} = 0 \\ R^{ij} &\equiv -\frac{h^2}{2} f^{ik} f^j_k - h^{-1} h^{;i;j} + \bar{R}^{ij} = 0 \end{aligned} \tag{3}$$

where

$$f_{ij} \equiv \omega_{j;i} - \omega_{i;j} \tag{4}$$

and $\bar{R}_{ij}$ is the Ricci tensor constructed from the spatial metric $\bar{g}_{ij}$.

In the case of two space dimensions, we can always choose isotropic coordinates such that

$$\bar{g}_{ij} = -e^{2u} \delta_{ij} \tag{5}$$

The gauge condition (2) then reduces to the flat space equation

$$\omega_{i,i} = 0 \tag{6}$$

which is solved by

$$\omega_i = \varepsilon_{ij}\, \partial_j \varphi \tag{7}$$

and the antisymmetric tensor field f_{ij} is given in terms of the scalar field φ by

$$f_{ij} = -\varepsilon_{ij}\,\Delta\varphi \tag{8}$$

(Δ being the two-dimensional Laplacian operator).

The second group of equations (3) may be written in isotropic coordinates as

$$\varepsilon_{ij}\tfrac{1}{2}h^{-1}\,e^{-2u}(e^{-2u}h^3\,\Delta\varphi)_{,j} = 0 \tag{9}$$

and are solved by

$$\Delta\varphi = \lambda\,e^{2u}h^{-3} \tag{10}$$

where λ is an arbitrary constant. Inserting this result in the first of equations (3), we obtain

$$\Delta h + \frac{\lambda^2}{2}\,e^{2u}h^{-3} = 0 \tag{11}$$

while the trace and the traceless part of the last group of equations (3), respectively, give

$$\Delta u - \frac{3\lambda^2}{4}\,e^{2u}h^{-4} = 0 \tag{12}$$

and

$$h_{;i;j} - \tfrac{1}{2}h_{;k}^{;k}g_{ij} = 0 \tag{13}$$

Introducing the complex variable $z = x + iy$, the equations (13) may be written as

$$\frac{\partial^2 h}{\partial z^2} = 2\frac{\partial u}{\partial z}\frac{\partial h}{\partial z} \tag{14}$$

which are solved by

$$\frac{\partial h}{\partial z}(z, \bar{z}) = \mu(\bar{z})\,e^{2u(z,\bar{z})} \tag{15}$$

where μ is an arbitrary function. We must distinguish between two cases:

(a) $\mu(\bar{z}) = 0$. Then h is constant and we may, without loss of generality, choose

$$h = 1 \tag{16}$$

The equation (11) can only be solved if $\lambda = 0$, which reduces equations (10) and (12) to

$$\begin{aligned} \Delta\varphi &= 0 \\ \Delta u &= 0 \end{aligned} \tag{17}$$

The multicenter solutions (up to an additive constant) of these equations

$$\begin{aligned} \varphi &= -\frac{\kappa}{2\pi}\sum_\alpha J_\alpha \ln|z-a_\alpha| \\ u &= -\frac{\kappa}{2\pi}\sum_\alpha m_\alpha \ln|z-a_\alpha| \end{aligned} \tag{18}$$

($\kappa = 8\pi G$ is the gravitational constant) lead to the metric

$$ds^2 = \left(dt + \frac{\kappa}{2\pi}\sum_\alpha J_\alpha \frac{\varepsilon_{ij}\, dx^i\, dx^j}{|\mathbf{r}-\mathbf{a}_\alpha|^2} \right)^2 - \prod_\alpha |\mathbf{r}-\mathbf{a}_\alpha|^{-(\kappa/\pi)m_\alpha}\, d\mathbf{x}^2 \tag{19}$$

representing a system of point particles of "masses" m_α (Deser et al., 1984; Clément, 1976, 1983) and "spins" J_α located at the points $\mathbf{a}_\alpha$. In the case of a single massless spinning particle at the origin, we recover the rotating solution of Deser et al. (1984):

$$ds^2 = \left(dt + \frac{\kappa}{2\pi} J\, d\theta \right)^2 - (dr^2 + r^2\, d\theta^2) \tag{20}$$

(b) $\mu(\bar{z}) \not\equiv 0$. If we define a new complex variable $Z = X + iY$ by

$$dZ = \mu^{-1}(z)\, dz \tag{21}$$

the equation (15) may be rewritten as

$$\frac{\partial h}{\partial Z} = |\mu|^2\, e^{2u} = e^{2U} \tag{22}$$

where the new spatial metric function U is defined by

$$e^{2u}\, dz\, d\bar{z} = e^{2U}\, dZ\, d\bar{Z} \tag{23}$$

It follows from (22) that h and U depend on the single variable X, so that equations (11) and (12) are both solved by

$$\frac{dh}{dX} - \frac{\lambda^2}{4} h^{-2} = \nu \tag{24}$$

where ν is another integration constant, and the equation (10) is solved by

$$\varphi(\mathbf{R}) = f(\mathbf{R}) + \frac{2}{\lambda}[h(X) - \nu X] \tag{25}$$

where f is an arbitrary harmonic function:

$$\Delta f(\mathbf{R}) = 0 \tag{26}$$

Defining a new time coordinate T by

$$dT = dt + \varepsilon_{ij} \frac{\partial f}{\partial X^j} dX^i \tag{27}$$

we finally obtain

$$ds^2 = h^2 \, dT^2 - \lambda \, dT \, dY - \nu \, dY^2 - \left(\nu + \frac{\lambda^2}{4h^2} \right)^{-1} dh^2 \tag{28}$$

The Newtonian potential h is infinite at spatial infinity, so that this solution does not admit a satisfactory physical interpretation [the metric (28) is further discussed in the Appendix].

3. GEODESIC MOTION IN THE FIELD OF A POINT PARTICLE

In this section we investigate the physical properties of the one-particle rotating solution

$$ds^2 = (dt + \omega \, d\theta)^2 - dr^2 - \alpha^2 r^2 \, d\theta^2 \tag{29}$$

where

$$\omega = \frac{\kappa}{2\pi} J, \qquad \alpha = 1 - \frac{\kappa}{2\pi} M \tag{30}$$

For $r = \infty$ to be an end point of the spatial sections of the metric (19), α must be positive (Deser et al., 1984; Clément, 1976, 1983), and we may assume $\omega \geq 0$.

We note that the metric (29) is the three-dimensional section of a particular case of four-dimensional cylindrical Gödel-like geometries (Gödel, 1949), defined by (Novello et al., 1982)

$$ds^2 = a^2[(dt + H(r) \, d\theta)^2 - dr^2 - R^2(r) \, d\theta^2 - dz^2] \tag{31}$$

In our case, $a = 1$, $H = \omega$, $R^2 = \alpha^2 r^2$.

As pointed out in Deser et al. (1984), the manifold of metric (29) contains closed timelike lines, for instance, the lines $t = t_0$, $r = r_0 < \omega/\alpha$. This does not necessarily lead to difficulties with causality, unless there are closed timelike geodesics. We shall see that the geodescis of the metric (29) are always open.

The first integrals of the geodesic equations of motion in the geometry (29) are

$$\dot{\theta}=\frac{l\gamma}{\alpha r^2}$$
$$\dot{t}=\frac{\gamma}{\beta}-\frac{\omega l\gamma}{\alpha r^2} \tag{32}$$
$$\dot{r}^2=\gamma^2-\frac{l^2\gamma^2}{r^2}$$

where the dot stands for the derivative with respect to an affine parameter τ, and l, β, γ are integration constants. The constant γ is an unessential normalization of the affine parameter, while β is the asymptotic velocity of the test particle; we shall assume for the discussion $\beta > 0$.

The constant l is related to the orbital angular momentum of the test particle. In the special case $l=0$, the motion of the particle is obviously uniform, at least until it hits the conical singularity $r=0$. In the general case $l \neq 0$, the test particle experiences apparent forces due to the coupling of its angular momentum with the mass [first equation (32)] and the spin [second equation (32)] of the singularity.

The equations of motion (32) are readily integrated, to give

$$r=\frac{|l|}{\cos\alpha(\theta-\theta_0)}$$
$$t-t_0=\frac{l}{\beta}\tan\alpha(\theta-\theta_0)-\omega(\theta-\theta_0) \tag{33}$$

Choosing $\theta_0=0$, we see that the open trajectory is continuously deflected from $\theta=-\pi/2\alpha$ to $\theta=+\pi/2\alpha$, so that the point mass M may be measured from the asymptotic deflection

$$\Delta\theta=-\left(\pi-\frac{\pi}{\alpha}\right)=\frac{\kappa M/2\pi}{1-\kappa M/2\pi} \tag{34}$$

To similarly single out the effect of the point spin J, we discuss the particular case $M=0$ ($\alpha=1$). In this case the test particle follows the straight line $x=|l|$ with the law of motion

$$t-t_0=\varepsilon(l)\left[\frac{y}{\beta}-\omega\arctan\left(\frac{y}{l}\right)\right] \tag{35}$$

For $l > 0$, the particle is accelerated as it nears the singularity to a maximum velocity

$$v_0 = \frac{\beta}{1 - \beta\omega/l} \tag{36}$$

for $y = 0$, then is decelerated as it goes away. The total time gain (with respect to a particle moving in empty space at the constant velocity β) is

$$\Delta t = \omega\pi \tag{37}$$

The process is reversed for $l < 0$, the time-gain being replaced by an equal time loss. So (assuming that these results can be carried over to quantum mechanics) a way to measure the point spin J would be to send a wavepacket toward the singularity, and measure the phase shift between the two left- and right-outgoing wavepackets, proportional to the time delay

$$2\Delta t = \kappa J \tag{38}$$

A peculiarity of this motion, which we shall discuss in the general case $\alpha \neq 1$, is that if

$$0 < l < \frac{\omega\beta}{\alpha} \tag{39}$$

then the velocity is accelerated to infinity as the particle nears a critical point of its trajectory. From there on, the particle appears to go backward in time, until a symmetrical critical point, where time resumes its normal course (Figure 1). This peculiarity, which also occurs in the case of the Gödel geometry does not, as stressed by Novello et al. (1982); represent a violation of causality. It is not directly observable because the test particle is free except for possible interactions at past and future time infinity. Moreover, what appears bizarre at the classical level can be explained at

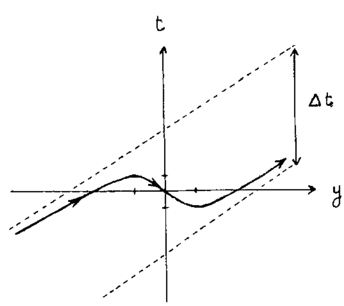

Fig. 1. A geodesic with $0 < l < \omega\beta/\alpha$.

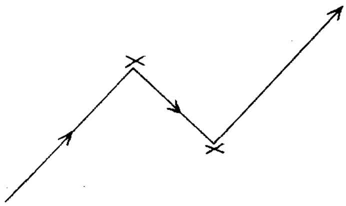

Fig. 2. The quantum equivalent of Fig. 1.

the quantum level in terms of pair annihilation and creation in the external metric field (Figure 2).

In the limit $\beta \to \infty$, the second equation (33) goes over to

$$t - t_0 = -\omega(\theta - \theta_0) \tag{40}$$

so that the corresponding open geodesics are contained in a finite time interval $\omega\pi/\alpha$. Of course these geodesics do not describe the motion of physical particles, for which $\beta \leq 1$.

APPENDIX

We further discuss here the stationary, cylindrical metric obtained at the end of Section 2 [case (b)]:

$$ds^2 = h^2\, dT^2 - \lambda\, dT\, dY - \nu\, dY^2 - \left(\nu + \frac{\lambda^2}{4h^2}\right)^{-1} dh^2 \tag{A1}$$

and its analytic continuation to $h^2 < 0$. This metric has the correct signature $(+--)$ as long as

$$\nu h^2 + \frac{\lambda^2}{4} > 0 \tag{A2}$$

We may distinguish three subcases according to the sign of ν:

(α) $\nu > 0$. The introduction of a new coordinate ξ such that

$$\nu^2 \xi^2 = \nu h^2 + \frac{\lambda^2}{4} \tag{A3}$$

enables us to rewrite the metric (A1) as

$$ds^2 = \nu\xi^2\, dT^2 - d\xi^2 - \nu\left(dY + \frac{\lambda}{2\nu}\, dT\right)^2 \tag{A4}$$

The obvious interpretation is that Y is an angular coordinate, so that the three-dimensional theory is effectively reduced, *à la* Kaluza-Klein, to two dimensions.

(β) $\nu=0$. The definition

$$\lambda\xi = h^2 \tag{A5}$$

leads to

$$ds^2 = \lambda\xi\left(dT - \frac{1}{2\xi}\,dY\right)^2 - \frac{1}{4\xi}\,dY^2 - d\xi^2 \tag{A6}$$

This metric stays perfectly regular ($\det g = -\lambda^2/4$) as ξ varies from $+\infty$ to $-\infty$. However the light cones, oriented along the T axis for $\xi \to +\infty$, gradually turn over as ξ decreases, until they are oriented along the Y axis for $\xi \to -\infty$.

(γ) $\nu<0$. The metric is again given by (A4), where Y is now to be interpreted as a time coordinate, and T as an angular space coordinate. Obviously, we recover in this case the one-particle rotating solution (29).

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